Design of Linear Regulators for Nonlinear Stochastic Systems

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The problem of regulating a nonlinear stochastic system with uncertain parameters and nonlinear noisy measurements is considered. The second-order terms in an expansion about the reference path are viewed as multiplicative disturbances in state and control. The linear estimation/control problem is solved by approximating these as wideband disturbances. The Certainty-Equivalence Principle and the Separation Theorem do not hold. The control and filter gains, found by solving a TPBVP, explicitly hedge against the parameter uncertainties and nonlinearity effects. Numerical results are presented for a combined guidance/navigation system for shuttle entry.

I. Introduction

In this paper a new technique for design of linear regulators for nonlinear, time-varying stochastic systems with certain parameters is developed. It is assumed that the desire is to follow a known reference path with minimum error. The available measurements are a set of known nonlinear functions of both the state and uncertain parameters. The performance index is a weighted sum of mean squared state deviations and estimation errors. A widely used solution technique is to transform to a linear variational problem by linearizing about a desired reference path in state space.[1,2] The familiar linear-quadratic-Gaussian (LQG) theory is then applied to derive a set of linear time-varying control feedback and filter gains. These gains are derived independently since, by the LQG assumption, both the Certainty-Equivalence Principle[2] and the Separation Theorem[3,4] hold. The LQG design works well if the effects of nonlinearities and parameter uncertainties are suitably small. However, since the LQG approach explicitly assumes both linearity and perfectly known parameters, it does not attempt to compensate for nonlinearities or parameter uncertainties.

The method of approach used here involves consideration of both the first- and second-order terms in the Taylor series expansion about a reference path. By inspection of the second-order terms, it is seen that nonlinearities and parameter uncertainties act as disturbances which multiply the state (path deviations), control, and estimation errors. These disturbances are modeled as wideband processes. With this assumption, the multiplicative uncertainties appear as white state-dependent noise (SDN) and control-dependent noise (CDN). The control of linear systems with SDN and CDN and quadratic cost function with perfect state information has been considered by several authors and a rather complete theory is available.[5-8] By assuming a linear filter structure, the theory yields a set of time-varying control feedback and filter gains, which can be computed *a priori*[9] by solving a two-point boundary-value problem (TPBVP). An especially interesting aspect of this approach is that control penalties are not necessarily required in the formulation of the optimization problem in order to obtain a solution. The problem is made nonsingular in the continuous case (which is not considered here; see Ref. 9) by the presence of CDN which adds convexity to the problem and acts essentially as additional control penalty. This provides a systematic and natural approach to determination of control weighting functions.

Presented as Paper 74-127 at the AIAA 12th Aerospace Sciences Meeting, Washington, D.C., January 30-February 1, 1974; submitted April 23, 1974; revision received February 7, 1975.

Index category: Navigation, Control and Guidance Theory.

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The method is applied to the design of an on-board guidance and navigation system for the space shuttle during the entry phase of flight. Since aerodynamic control is used throughout the entry phase, this is an example of a stochastic control problem without a natural penalty on control activity. The control limitations are more naturally given in terms of maximum actuator rate and amplitude limits and, perhaps, pilot comfort. Over much of the flight these constraints will not be active and the control can be considered cost-free. The control problem is made complex by the presence of uncertainties in the aerodynamic coefficients and atmospheric density, which appear as multiplicative disturbances.

Statistical data were generated by solving the TPBVP iteratively, using successive approximations. The results indicate that both filter and control gains reflect the uncertainty in the parameters. Density uncertainty produces hedging during initial atmospheric penetration, while aerodynamic coefficient uncertainties produce hedging near the end of the entry.

II. Problem Formulation

A. Nonlinear System Description

Consider the system which evolves according to the vector nonlinear recursive equation

$$x_i = f_i(x_{i-1}, u_{i-1}, \alpha_{i-1}, i-1) \tag{1}$$

with the initial condition x_o a random vector with mean $\hat{x}_o$, and covariance P_o. Here, x is the n-dimensional state, u is the p-dimensional control, and i is the time index. The argument α is used to denote dependence on a set of r unknown parameters with known first and second moments. It is assumed that no elements of α will be estimated during the regulation process. The function f is twice continuously differentiable in x_i, u_i, and α_i for all i.

The objective is to keep the state x_i close to a known reference or nominal path x_i^N described by

$$x_i^N = f_i(x_{i-1}^N, u_{i-1}^N, \alpha_{i-1}^N, i-1) \tag{2}$$

with known initial condition x_0^N. The values of u_j^N, α_j^N for $j=1,\ldots, i-1$ are assumed known so that the nominal path is completely deterministic. The superscript N is used throughout to denote the nominal value of a parameter. The deviation from the nominal is $\delta x_i = x_i - x_i^N$ and is, in general, nonzero if either the initial conditions, the controls, or the parameters are off-nominal.

The available measurements are a set of m nonlinear functions of the state and the parameters given by

$$z_i = h_i(x_i, \alpha_i, i) \tag{3}$$

Note that α_i may be used to model additive noise in either Eq. (1) or Eq. (3). The state estimate is assumed to evolve according to

$$\hat{x}_i = f_i(\hat{x}_{i-1}, u_{i-1}, \alpha_{i-1}^N, i-1)$$
$$+K_i[h_i(x_i, \alpha_i, i) - h_i(\hat{x}_i, \alpha_i^N, i)] \tag{4}$$

with K_i the filter gain matrix and h_i a twice continuously differentiable function of x_i and α_i for all i. The estimated dynamics $\hat{f}_i \triangleq f_i(\hat{x}_{i-1}, u_{i-1}, \alpha_{i-1}, \alpha_{i-1}^N, i-1)$ are defined in terms of estimated state $\hat{x}_i$, the known control u_i and the nominal parameter vector α_i^N

B. Expansion to Second-Order and SDN, CDN Approximations

In order to formulate the linear regulator problem the actual state and the estimated state are expanded to second order about their nominal values. For the moment assume $K_i = 0$. Let f^N† denote the right-hand side of Eq. (2) and denote partial differentiation by a subscript. Then, on an actual trajectory, the expansion is symbolically of the form

$$x_i = f^N + f_x{}^N\delta x_{i-1} + f_u{}^N\delta u_{i-1} + f_\alpha{}^N\delta\alpha_{i-1}$$
$$+ \tfrac{1}{2}f_{xx}^N\,\delta x_{i-1}\delta x_{i-1} + \tfrac{1}{2}f_{uu}^N\,\delta u_{i-1}\delta u_{i-1} + \tfrac{1}{2}f_{\alpha\alpha}^N\delta\alpha_{i-1}\delta\alpha_{i-1}$$
$$+ f_{\alpha x}^N\delta\alpha_{i-1}\delta x_{i-1} + f_{ux}^N\delta u_{i-1}\delta x_{i-1} + f_{\alpha u}{}^N\delta\alpha_{i-1}\delta u_{i-1} \tag{5}$$

where $\delta u_i = u_i - u_i^N$ $\delta\alpha_i = \alpha_i - \alpha_i^N$ and all partial derivatives are evaluated along the nominal path. The second partial derivatives of f are third-order tensors and Eq. (5), thus, is not a vector-matrix equation. The notation convention used is that the components of $f_x\delta x$ and $f_{xx}\delta x\delta x$ are given by, upon dropping the time index

$$(f_x{}^N\delta x)_j \triangleq \sum_{k=1}^{n} \frac{\partial f_j{}^N}{\partial x_k}\delta x_k \triangleq \sum_{k=1}^{n} f_{x_{jk}}^N\delta x_k$$

$$(f_x{}^N\delta x)_j \triangleq \sum_{k=1}^{n} \frac{\partial f_j{}^N}{\partial x_k}\delta x_k \triangleq \sum_{k=1}^{n} f_{x_{jk}}^N\delta x_k$$

$$(f_{xx}^N\,\delta x\,\delta x)_j \triangleq \sum_{k,\ell=1}^{n} \frac{\partial^2 f_j{}^N}{\partial x_k \partial x_\ell}\delta x_k\delta x_\ell = \sum_{k,\ell=1}^{n} f_{xx_{jk\ell}}^N\,\delta x_k\delta x_\ell$$

The components of the remaining terms are defined in a similar manner.

By rearranging Eq. (5) the variational equation may be written in the form

$$\delta x_i = f - f^N$$
$$= f_x{}^N\delta x_{i-1} + f_u{}^N\delta u_{i-1} + f_\alpha{}^N\delta\alpha_{i-1} + \tfrac{1}{2}f_{\alpha\alpha}^N\,\delta\alpha_{i-1}\delta\alpha_{i-1}$$
$$+ [\tfrac{1}{2}f_{xx}^N\,\delta x_{i-1} + f_{\alpha x}^N\,\delta\alpha_{i-1}]\delta x_{i-1} + f_{xu}^N\delta x_{i-1}\delta u_{i-1}$$
$$+ [f_{\alpha u}^N\,\delta\alpha_{i-1} + \tfrac{1}{2}f_{uu}^N\,\delta u_{i-1}]\delta u_{i-1} \tag{6}$$

The form of this equation suggests that the coefficients of δx and δu within brackets may be modeled as state-dependent noise (SDN) and control-dependent noise (CDN), respectively. However, the role of the term $f_{xu}^N\,\delta x_{i-1}\,\delta u_{i-1}$ is not clear. Before making these definitions, it is more appropriate to consider the estimator error equation. This leads to a redefinition of the SDN and CDN terms.

†The subscript i on f and h is, for convenience, dropped in the sequel.

Expanding the estimator equation (4) about the nominal with $K_i = 0$ gives, to second-order

$$\delta\hat{x} = \hat{x}_i - x_i^N$$
$$= f_x{}^N\delta\hat{x}_{i-1} + f_u{}^N\delta u_{i-1} + \tfrac{1}{2}f_{xx}^N\delta\hat{x}_{i-1}\delta\hat{x}_{i-1}$$
$$+ [f_{xu}^N\delta\hat{x}_{i-1} + \tfrac{1}{2}f_{uu}^N\delta u_{i-1}]\,\delta u_{i-1} \tag{7}$$

Now using $e = \delta\hat{x} - \delta x$ and subtracting Eq. (6) from Eq. (7) produces the error equation

$$e_i = \delta\hat{x}_i - \delta x_i$$
$$= f_x{}^N e_{i-1} - f_\alpha{}^N\delta\alpha_{i-1} - \tfrac{1}{2}f_{\alpha\alpha}^N\,\delta\alpha_{i-1}\delta\alpha_{i-1}$$
$$+ [f_{xx}^N\,e_{i-1} - f_{\alpha x}^N\delta\alpha_{i-1}]\delta x_{i-1}$$
$$- [f_{\alpha u}^N\,\delta\alpha_{i-1} - f_{xu}^N\,e_{i-1}]\delta u_{i-1} + \tfrac{1}{2}f_{xx}^N\,e_{i-1}e_{i-1} \tag{8}$$

In order to cast Eq. (8) into an appropriate form with SDN and CDN, several assumptions must be made. The term $f_\alpha^N\delta\alpha$ is modeled by a white noise sequence denoted by w_{i-1}. The second-order terms $f_{\alpha\alpha}^N\delta\alpha\delta\alpha/2$ and $f_{xx}^N ee/2$ are assumed small with respect to $f_\alpha^N\delta\alpha$ and dropped.‡ Define

$$\Delta\Phi_i \triangleq -f_{xx}^N\,e_{i-1} + f_{\alpha x}^N\,\delta\alpha_{i-1} \tag{9}$$

$$\Delta G_i \triangleq F_{\alpha u}^N\,\delta\alpha_{i-1} - f_{xu}^N\,e_{i-1} \tag{10}$$

The variations of the $n\times n$ matrix $\Delta\Phi$ are assumed to be wideband with respect to the variations of δx. Similarly, the variations of the $n\times p$ matrix ΔG are assumed to be wideband with respect to the variations of δu. If it is further assumed that the elements of $\Delta\Phi$ and ΔG are mutually independent and independent of δx and δu, then $\Delta\Phi$ and ΔG are SDN and CDN matrices, respectively. This assumption can be relaxed as long as the dynamic equations still remain a Markov process.

The use of white noise to model multiplicative uncertainty here is analogous to the familiar use of additive white noise to model additive disturbances which in reality may be highly correlated in time or even constant biases. An alternative to using additive white noise for modeling additive disturbances is augmentation of the state to include the disturbances as state elements. It may be possible in that case to retain the linear character of the dynamics. With multiplicative noise this is not possible; time-correlated multiplicative noise leads inevitably to nonlinear dynamics and generally intractable problems. The advantages of using white SDN and CDN are that 1) an analytical solution may be obtained, and 2) the disturbances are placed where they actually occur in the system. This is of importance since additive noise and multiplicative noise affect the dynamics in a much different manner.

Now consider the estimator equation (4) with $K_i \neq 0$. The measurement h and the estimate $\hat{h} \triangleq h(\hat{x}_i, \alpha_i^N, i)$ are expanded to second-order about their nominal values as

$$h = h^N + h_x^N\delta x_i + h_\alpha^N\delta\alpha_i + \tfrac{1}{2}h_{xx}^N\delta x_i\delta x_i$$
$$+ h_{\alpha x}^N\delta\alpha_i\delta x_i + \tfrac{1}{2}h_{\alpha\alpha}^N\delta\alpha_i\delta\alpha_i \tag{11}$$

$$\hat{h} = h^N + h_x{}^N\delta\hat{x}_i + \tfrac{1}{2}h_{xx}^N\delta\hat{x}_i\delta\hat{x}_i \tag{12}$$

From these, the measurement residual is found to be

$$h - \hat{h} = -h_x{}^N e_i + h_\alpha^N\,\delta\alpha_i + \tfrac{1}{2}h_{\alpha\alpha}^N\delta\alpha_i\delta\alpha_i$$
$$+ [h_{\alpha x}^N\delta\alpha_i - h_{xx}^N\,e_i]\delta x_i - \tfrac{1}{2}h_{xx}^N e_i e_i \tag{13}$$

‡These quadratic terms could be approximated by their mean value, if necessary, since the second moments of e_i and $\delta\alpha_{i-1}$ are assumed approximately known.

Define the $m \times n$ matrix ΔH_i by

$$\Delta H_i \triangleq h^N_{\alpha x}\delta\alpha_i - h^N_{xx} e_i \tag{14}$$

If the variations of ΔH are assumed wideband with respect to variations in δx, and if ΔH is assumed independent of δx and e, then ΔH is in the form of white measurement SDN. The term $h^N_\alpha \delta\alpha$ is modeled by a white noise sequence denoted by r_{i-1}.

C. Linearized System Description

Based on the preceding discussion, the original nonlinear system is now

$$x_i = (\Phi_i + \Delta\Phi_i)x_{i-1} + (G_i + \Delta G_i)u_{i-1} + w_i \tag{15}$$

$$z_i = (H_i + \Delta H_i)x_i + r_i \tag{16}$$

where x,u are now *variations* from x^N, u^N and where $\Phi_i = f^N_x$, $G_i = f^N_u$, $H_i = h^N_x$ all evaluated at stage i. The initial condition x_o has known first and second moments and w_i, r_i are mutally uncorrelated vector white sequences with zero mean and covariances $E(w_i w_j^T) = W_i\delta_{ij}$, $E(r_i r_j^T) = R_i\delta_{ij}$ where the operation $E(.)$ denotes expectation over all random variables and δ_{ij} is the Kronecker delta function. The SDN in the dynamics is modeled by the $n \times n$ matrix $\Delta\Phi_i$, whose elements are zero-mean white sequences with covariance $E[(\Delta\Phi_i)_{jk}(\Delta\Phi_l)_{mn}] = (V_i)_{jkmn}\delta_{il}$. Here, $(\Delta\Phi_i)_{jk}$ is the element in the j^{th} row and the k^{th} column of $\Delta\Phi_i$ and V_i is a 4$^{\text{th}}$ order tensor of n^4 components. The CDN is modeled by the $n \times p$ matrix ΔG_i, whose elements are zero-mean white sequences with covariance $E[(\Delta G_i)_{k\ell}(\Delta G_j)_{mn})] = U_{k\ell mn}\delta_{ij}$. With the assumption that $\Delta\Phi_i$, ΔG_i, w_i are white, Eq. (15) forms a Markov process. To simplify the algebra in the following analysis, it is assumed that the above noise terms are mutually uncorrelated. The extension is straightforward. The SDN in the measurements is modeled by the $m \times n$ matrix ΔH_i, whose elements are zero-mean white sequences with covariance

$$E[(\Delta H_i)_{jk}(\Delta H_l)_{mn}] = (Q_i)_{jkmn}\delta_{il}$$

The control u_i is restricted to a linear function of the state estimate $\hat{x}_i$ as

$$u_{i-1} = L_{i-1}\hat{x}_{i-1} \tag{17}$$

where $\hat{x}_i$ is the estimated variation from x^N, obtained from the data set $[z_o, z_i, \ldots, z_i]$ using Eq. (4).

C. Covariance Equations and Performance Criterion

Substituting Eq. (17) into Eq. (15), the closed-loop transition matrix is defined as $\Phi_i^c \triangleq \Phi_i + G_i L_{i-1}$. With these assumptions, the form of the linear, unbiased estimator is

$$\hat{x}_i = (I - K_i H_i)\Phi_i^c \hat{x}_{i-1} + K_i z_i \tag{18}$$

The equations for propagation of the state and error $e = \hat{x} - x$ are

$$x_i = (\Phi_i^c + \Delta\Phi_i)x_{i-1} + G_i L_{i-1} e_{i-1} + \Delta G_i L_{i-1}\hat{x}_{i-1} + w_i \tag{19}$$

$$e'_i = \Phi_i e_{i-1} - \Delta\Phi_i x_{i-1} - \Delta G_i L_{i-1}\hat{x}_{i-1} - w_i \tag{20}$$

$$e_i = (I - K_i H_i)e'_i + + K_i(\Delta H_i x_i + r_i) \tag{21}$$

Now define the following covariances

$$P_i = E(e_i e_i^T),\; C_i = E(e_i x_i^T)$$

$$X_i = E(x_i x_i^T),\; \hat{X}_i = E(\hat{x}_i \hat{x}_i^T)$$

§The prime denotes association with an estimate prior to measurement incorporation.

Then the recursion relations for propagation of the covariances are

$$P'_i = \Phi_i P_{i-1}\Phi_i^T + \Delta(X_{i-1}) + \Omega_i(L_{i-1}\hat{X}_{i-1}L_{i-1}^T) + W_i \tag{22}$$

$$P_i = (I - K_i H_i)P'_i(I - K_i H_i)^T + K_i[\Gamma_i(X_i) + R_i]K_i^T \tag{23}$$

$$C_i = (I - K_i H_i)[\Phi_i(C_{i-1}\Phi_i^{cT} + P_{i-1}L_{i-1}{}^T G_i^T) - \Delta_i(X_{i-1}) - \Omega_i(L_{i-1}\hat{X}_{I-1}L_{i-1}{}^T) - W_i] \tag{24}$$

$$X_i = \Phi_i^c X_{i-1}\Phi_i^{cT} + G_i L_{i-1}C_{i-1}\Phi_i^{cT} + \Phi_i^c C_{i-1}{}^T L_{i-1}{}^T G_i^T + W_i + G_i L_i P_{i-1}{}^T G_i^T + \Delta_i(X_{i-1}) + \Omega_i(L_{i-1}\hat{X}_{i-1}L_{i-1}{}^T) \tag{25}$$

where

$$\Delta_i(\cdot) \triangleq E[\Delta\Phi_i(\cdot)\Delta\Phi_i^T],\; \Gamma_i(\cdot) \triangleq E[\Delta H_i(\cdot)\Delta H_i^T],$$

$$\Omega_i(\cdot) \triangleq E[\Delta G_i(\cdot)\Delta G_i^T]$$

$$\hat{X}_{i-1} = X_{i-1} + C_{i-1} + C_{i-1}{}^T + P_{i-1} \tag{26}$$

The cost function to be minimized is the quadratic form

$$J(K,L,N) = E\sum_{i=1}^{N}\{e_i^T W_{p_i} e_i + x_i^T W_{x_i} x_i + u_{i-1}{}^T W_{u_{i-1}} u_{i-1}\} \tag{27}$$

where W_{pi}, W_{xi}, W_{ui-1} are non-negative-definite symmetric weighting matrices, for all i. The arguments K, L are used to denote gain sequences $\{K_1, K_2, \ldots, K_N\}$ and $\{L_0, L_1, \ldots L_{N-1}\}$.

III. Calculation of Optimal Filter and Control Gains

By substituting P, X, and C into Eq. (27), the problem is reduced to a deterministic optimization problem subject to the constraint equations (22-25) and known initial conditions P_0, C_0, X_0. This is solved as follows[10,11]: Denote the right-hand side of Eqs. (23-25) by ψ_{pi-1}, ψ_{ci-1}, ψ_{xi-1}, respectively, and adjoin these as constraints to $J(K, L, N)$ with a set of Lagrange multiplier matrices $\wedge_{pi}$, $\wedge_{c_i}$, $\wedge_{x_i}$. Then the augmented cost is

$$\bar{J}(K,L,N) = J(K,L,N) + \sum_{i=1}^{N} tr\{\wedge_{p_i}(\Psi_{p_{i-1}} - P_i) + 2\wedge_{c_i}^T(\Psi_{c_{i-1}} - C_i) + \wedge_{x_i}(\Psi_{x_{i-1}} - X_i)\} \tag{28}$$

Stationary conditions are found by setting derivatives of $\bar{J}$ with respect to K_i, P_i, X_i, L_i to zero. If this is done, the following conditions for optimality are obtained. The optimal gain matrices are

$$K_i = P'_i H_i^T[H_i P'_i H_i^T + \Gamma_i(X_i) + R_i]^{-1} \tag{29}$$

$$L_{i-1} = -[G_i^T(S_i + \wedge_{x_i})G_i + W_{u_{i-1}} + \bar{\Omega}_i(S_i + \wedge_{x_i} + Y_i^T\wedge_{pi}Y_i)]^{-1}G_i^T(S_i + \wedge_{x_i})\Phi_i \tag{30}$$

and the Lagrange multipliers are propagated backward according to

$$\wedge_{p_{i-1}} = \Phi_i^T Y_i^T \wedge_{p_i} Y_i \Phi_i + W_{p_{i-1}} + L_{i-1}{}^T\{G_i^T(S_i + \wedge_{x_i})G_i + W_{u_{i-1}} + \bar{\Omega}_i(S_i + \wedge_{x_i} + Y_i^T\wedge_{p_i}Y_i)\}L_{i-1} \tag{31a}$$

$$\Lambda_{p_N} = W_{p_N} \tag{31b}$$

$$\Lambda_{x_{i-1}} = \Phi_i^{cT}(S_i + \Lambda_{x_i})\Phi_i^C + W_{x_{i-1}}$$
$$+ \bar{\Delta}_i(S_i + \Lambda_{x_i} + Y_i^T \Lambda_{p_i} Y_i)$$
$$+ L_{i-1}^T [W_{u_{i-1}} + \bar{\Omega}_i(S_i + \Lambda_{x_i} + Y_i^T \Lambda_{p_i} Y_i)] L_{i-1} \tag{32a}$$

$$\Lambda_{x_N} = W_{x_N} \tag{32b}$$

where $Y_i \triangleq I - K_i H_i$; $S_i \triangleq \bar{\Gamma}_i(K_i^T \Lambda_{p_i} K_i)$,

with the linear operators $\bar{\Gamma}$, $\bar{\Omega}$, $\bar{\Delta}$ defined by

$$\bar{\Gamma}_i(\cdot) \triangleq E[\Delta H_i^T(\cdot)\Delta H_i]; \; \bar{\Omega}_i(\cdot) \triangleq E[\Delta G_i^T(\cdot)\Delta G_i]$$

$$\bar{\Delta}_i(\cdot) \triangleq E[\Delta \Phi^T(\cdot)\Delta \Phi]$$

In addition, it follows that $\hat{C}_i \triangleq E[e_i \hat{x}_i^T] = 0$, so that the estimate and the error are uncorrelated. Equations (22, 23, 25, and 29-32) form a two-point boundary-value problem for the extremal solution.

Sufficient conditions for optimality may be obtained from the second partial derivatives of J with respect to K and L. The results is that sufficient conditions for a strong minimum are

1) $G_i^T(S_i + \Lambda_{x_i})G_i + \bar{\Omega}_i(S_i + \Lambda_{x_i}) + \bar{\Omega}_i(Y_i^T \Lambda_{p_i} Y_i) + W_{u_{i-1}} > 0$

2) $\hat{X}_i > 0$

3) $H_i P'_i H_i^T + \Gamma_i(X_i) + R_i > 0$

The solution of this problem is an example of cautious control[12] where the control serves the purposes of both control and improving knowledge of the state. In this case, the control influence on the filter performance enters directly through the SDN and CDN terms in Eqs. (22), and (29). The desire from both a control and estimation standpoint is to keep the actual state of the system as close as possible to the origin. The effect of control on filter performance may be brought out more explicitly by considering the pure estimation problem with no CDN. By setting $\Lambda_{x_i} = 0$, $W_{u_{i-1}} = 0$ in Eq. (30), the control gain L_{i-1} is, in general, still well-defined, due to the presence of SDN. Note that, if $W_{x_i} = 0$ for all i, $\Lambda_{x_i} \neq 0$ with SDN but is identically zero with no SDN. For the case without SDN or CDN, the solution reduces to that of the familiar linear quadratic Gaussian (LQG) problem, which requires the solution of two uncoupled one-point boundary-value problems. The optimal deterministic controller, which is linear in the present state estimate, is operated in cascade with a Kalman filter. The control and estimation problems completely separate in this case and the gains are determined independently.[13,14]

IV. Computation of State- and Control-Dependent Noises

In order to apply the theory of Sec. III, the covariance matrices involving the SDN and CDN must be found. In what follows, the square matrix M has dimensions consistent with the arguments of the various covariance matrices. From Eq. (9) and Eq. (26)

$$\Delta(M)_{ij} = E \sum_{k,\ell=1}^{n} \{ (f_{xx}^N e - f_{\alpha x}^N \delta\alpha)_{ik} M_{k\ell} (f_{xx}^N e - f_{\alpha x}^N \delta\alpha)_{\ell j} \}$$

$$= E \sum_{k,\ell=1}^{n} \sum_{m,o=1}^{n} \{ f_{xx_{imk}}^N f_{xx_{\ell oj}}^N e_m e_o M_{k\ell} \}$$

$$+ E \sum_{k,\ell=1}^{n} \sum_{m,o=1}^{r} \{ f_{\alpha x_{imk}}^N f_{\alpha x_{\ell oj}}^N \delta\alpha_m \delta\alpha_o M_{k\ell} \}$$

where

$$f_{xx_{imk}}^N = \frac{\partial^2 f_i}{\partial x_m \partial x_k}, \text{ etc.}; \; f_{\alpha x_{imk}}^N = \frac{\partial^2 f_i}{\partial \alpha_m \partial x_k}, \text{ etc.}$$

and $\delta\alpha$ and e have been assumed mutually independent. Let $A = E(\delta\alpha\, \delta\alpha^T)$. Then

$$\Delta(M)_{ij} = \sum_{k,\ell,m,o=1}^{n} f_{xx_{imk}}^N f_{xx_{\ell oj}}^N P_{mo} M_{k\ell}$$
$$+ \sum_{k,\ell=1}^{n} \sum_{m,o=1}^{r} f_{\alpha x_{imk}}^N f_{\alpha x_{\ell oj}}^N A_{mo} M_{k\ell} \qquad i,j = 1,2,\ldots,n$$

The matrices $\bar{\Delta}$, Ω, $\bar{\Omega}$, Γ, $\bar{\Gamma}$ are computed in a similar manner.[15]

V. Application to Design of an Entry Guidance and Navigation System

A problem of current interest is the design of on-board guidance and navigation systems for the NASA space shuttle vehicle. A particularly critical phase of the mission is the entry portion, where tight trajectory control is desired to meet accuracy requirements prior to the final approach phase. In addition, tight path control is necessary to minimize excess heating on off-nominal trajectories. The portion of entry flight considered here is between 400,000 and 100,000 ft alt where most of the kinetic energy is lost and most of the total heat load is absorbed.

During entry, measurements of specific force (nongravitational acceleration) are available from a set of accelerometers with orthogonal input axes which are mounted on an inertially-stabilized platform. No other information is assumed available for use in navigation. The assumed controls are angle-of-attack and roll angle which are, in turn, related in a known manner to aerodynamic control surface deflections. This is a good example of a problem without natural control penalties, since the essential control constraints are the physical limitations of control surface deflection amplitudes and rates. A large control deflection is intrinsically no more undesirable than a small control deflection if within the physical limits.

The basic problem is to maintain the vehicle close to the desired path in the presence of initial condition errors, trajectory disturbances, and measurement errors. The problem is made more complex by the presence of uncertainties in the aerodynamic coefficients and atmospheric density, which can have a significant effect on system performance. The desire is to somehow account for these uncertainties without estimating them. The geometry for the entry problem is depicted in Fig. 1. The assumed orientation of the accelerometer input axes is along x_P and y_P, which denote the platform frame. Both accelerometers are misaligned from the assumed orientation by the angle θ_c. Accelerometer outputs thus measure specific force along the x_s and y_s axes. In order to simplify the analysis several assumptions were made. Flight is made over a nonrotating spherical earth using an inverse-square gravity model. An exponential density model with scale height $h_s = 23{,}500$ ft was used. Only the in-plane portion of the flight was considered. This is a reasonable assumption since one possible method of controlling out-of-plane motion is to use roll switching (i.e., change the sign of roll angle while maintaining roll angle magnitude).[16,17] Flight was assumed to take place in the equatorial plane. With these assumptions, the state is given by $x = (h, \Omega, v, \gamma)^T$ with h the altitude in feet, Ω longitude in radians, v the velocity magnitude in fps and γ the flight path angle in radians, measured from the local horizontal. The available controls are the angle-of-attack ($\bar{\alpha}$) and roll angle (ϕ): $u^T = [\bar{\alpha}, \phi]$.

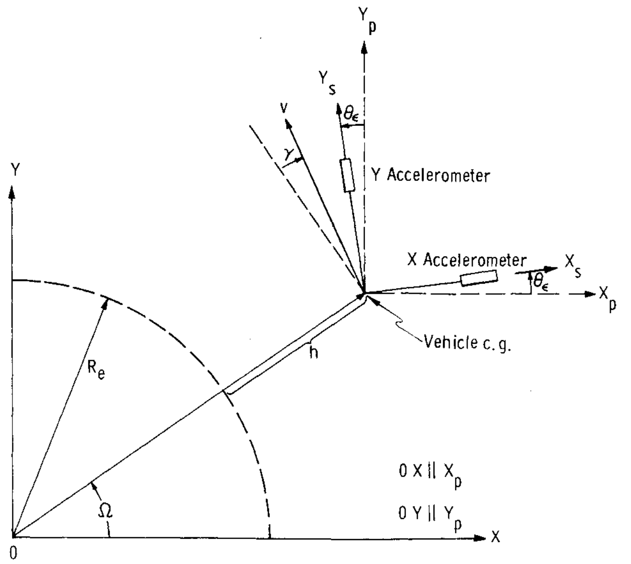

Fig. 1 Geometry for entry problem.

A. Dynamic System

The equations of motion are

$$h = v\sin\gamma \tag{33}$$

$$\dot{\Omega} = (v\cos\gamma / R_e + h) \tag{34}$$

$$v = -D - [\mu \sin\gamma / (R_e + h)^2] \tag{35}$$

$$\dot{\gamma} = \frac{L\cos\phi}{v} + [\frac{v}{R_e + h} - \frac{\mu}{v(R_e + h)^2}]\cos\gamma \tag{36}$$

where R_e is the earth's radius and μ is the gravitational constant. The parameters D and L are the aerodynamic specific drag and lift accelerations given by $D = \frac{1}{2}\rho v^2 (C_D(\bar{\alpha})A/m)$, $L = \frac{1}{2}\rho v^2 (C_L(\bar{\alpha})A/m)$ where ρ is the atmospheric density given by $\rho = \rho_0 e^{-h/h_s}$. The measurements are the accelerometer outputs given by

$$z_1 = a_X = L\cos\varphi\cos(\Omega + \theta_\epsilon - \gamma) + D\sin(\Omega + \theta_\epsilon - \gamma) + a_{BX}$$

$$z_2 = a_Y = L\cos\varphi\sin(\Omega + \theta_\epsilon - \gamma) - D\cos(\Omega + \theta_\epsilon - \gamma) + a_{BY}$$

where a_{BX}, a_{BY} are the x and y accelerometer errors.

The vehicle parameters and nominal trajectory data used in the sequel were taken from Ref. 18. The parameter A is the aerodynamic reference area and m is the fixed vehicle mass. Nominal parameters values used were $R_e = 20{,}925{,}722$ ft, $\rho_0 = 0.0027$ slugs/ft^3, $A = 6474$ ft^2, $W = 268{,}000$ lb. The lift and drag coefficients, obtained by fitting aerodynamic data for the North American Rockwell 161-C vehicle, were

$$C_D(\bar{\alpha}) = C_{D0} + C_{DI}\sin^3\bar{\alpha}, \quad C_L(\bar{\alpha}) = C_{LI}\sin^2\bar{\alpha}\cos\bar{\alpha}$$

with $C_{D0} = 0.04$, $C_{DI} = 2.2$, $C_{LI} = 2.44$.

The parameter vector α consists of the sea-level density ρ_o, platform alignment error θ_ϵ, and the aerodynamic coefficients: $\alpha = [\rho_0 \theta_\epsilon \; C_{D0} \; C_{DI} \; C_{LI}]^T$. Based on physical considerations these were felt to represent the most significant system uncertainties. Other parameters such as reference area or mass can easily be included. Winds can also be modeled as a parameter uncertainty. The nominal entry trajectory was designed to minimize total thermal protection system weight. The initial conditions were $h_o = 400{,}000$ ft, $\Omega_o = 0$, $\gamma_o = -1.58°$, and $v_0 = 25{,}900$ fps.

B. Numerical Characterization

The optimal angle-of-attack and roll angle histories given in Ref. 18 were fitted to generate the nominal trajectory. The equations of motion were integrated using a fourth-order Runge-Kutta integration routine with an 8 sec time step. The results are shown in Figs. 2a-d for a total elapsed time of 2000 sec. The terminal conditions were range = 5566 nautical miles, $h_f = 109{,}566$ ft, $v_f = 3130$ ft/sec, $\gamma_f = -3.62°$. The nominal trajectory begins with an initial penetration to an altitude of 260,000 ft at 280 sec. At this point, the heat rate has built up to 63 BTU/ft^2/sec. From 280-500 sec. there is a slight pull-up maneuver, after which the altitude decreases steadily but with a slight phugoid oscillation. Heat is spread out over a period of time to avoid heat peaks.

The initial value of the state covariance matrix and the estimation error covariance matrix were set equal to each other

$$X_o = P_o = \begin{bmatrix} 6.611\times10^6 & -0.4689 & -15957 & 0.1830 \\ & 7.509\times10^{-8} & 1.163\times10^{-3} & 1.102\times10^{-4} \\ & & 40.58 & -3.462\times10^{-4} \\ \text{symmetric} & & & 1.375\times10^{-8} \end{bmatrix}$$

and are typical of expected uncertainties at entry interface. Note that setting $X_0 = P_0$ implies $\hat{X}_0 = 0$, which in turn, implies perfect guidance execution for the deorbit burn made prior to entry. The rms parameter errors used were density 30%; platform alignment, 0.006 rad; aerodynamic coefficients, 20%. In addition, an rms accelerometer error of 0.001 fps^2 was assumed.

The values of the multiplicative noise matrices were multiplied in the program by a scalar factor η_s for SDN and η_c for CDN. For example, $\Delta(X)$ was replaced by $\eta_s\Delta(X)$. The need for the modification arises for two reasons. First, since the assumption of white noise is only approximate, a value of η other than unity should give the best model for the actual system uncertainties. Secondly, the value of η may be used as a design parameter, especially for the CDN. This is analogous to the use of the weighting matrices in the original cost function as design parameters. In computing the multiplicative noise matrices, only the terms involving parameter uncertainties were considered. This is a reasonable assumption since the parameter variations have a more significant effect on the problem than the second order estimation errors. The partial derivatives $f_x^N, f_u^N, f_\alpha^N, f_{\alpha x}^N, f_{\alpha u}^N$ required in the solution correspond to the discrete problem, and were obtained by integration of the variational equations of the continuous dynamics, Eqs. (33-36). Note that f_x^N is the transition matrix for the variational equation.

Since the basic objective is to follow the nominal path as closely as possible, the estimation error weighting matrices $\{W_{P_i}\}$ were set to zero over the entire path. Estimation is used only to improve tracking accuracy. The state deviation weighting matrices were kept constant at each discrete time point with a value

$$W_x = 0.01 \begin{bmatrix} (\frac{1}{1000})^2 & & 0 & \\ & (60)^2 & & \\ & & (\frac{1}{100})^2 & \\ 0 & & & (60)^2 \end{bmatrix}$$

which corresponds, approximately, to equal weightings assigned to 1000 ft alt deviation, 60 naut mile downrange deviation, 100 fps speed deviation and 1° flight path angle deviation. The weightings on range and speed deviations are somewhat light since they are basically limited by estimation accuracy. Tightening of range and speed control does not improve system performance since, with the present values, the rms range and speed deviations are only slightly greater than the rms estimation errors. The control weighting matrix used was constant with a value

¶Obtained from B. A. Kriegsman of C. S. Draper Laboratory, Inc., Cambridge, Mass.

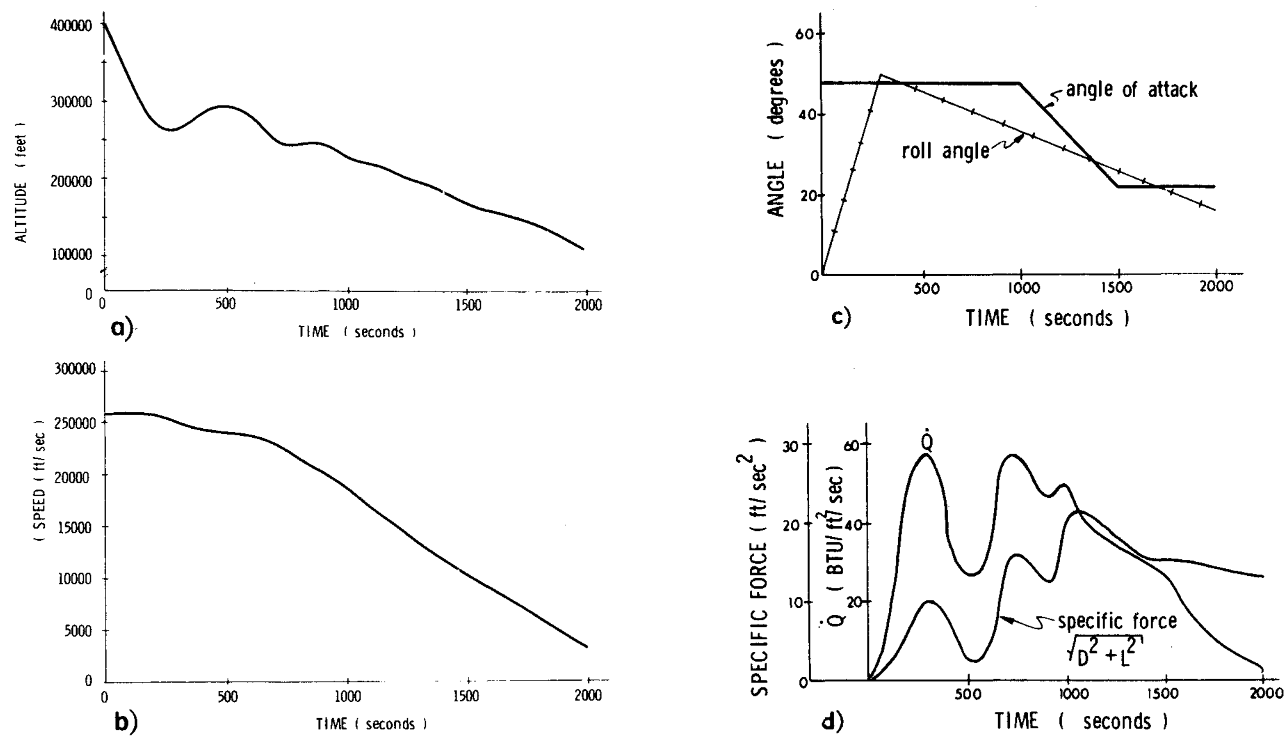

Fig. 2 a) Nominal altitude profile, b) nominal speed profile, c) nominal control histories, d) nominal heat rate and specific force.

$$W_u = W_{u_o}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The method of successive approximation was used to solve the two-point boundary-value problem. Initially, the covariances were propagated forward from the known initial conditions using no control. The transition matrix ϕ and filter gain matrix K were stored on the forward pass for use on the backward pass. The Lagrange multiplier Λ_x and Λ_p were then propagated backward starting from their known terminal conditions, using the stored values of ϕ and K. The control gain matrix L was stored on the backward pass for use on the succeeding forward pass. The nominal state and partial derivatives were computed and stored on the first forward pass for use in all succeeding forward and backward passes. The convergence of the iterations was checked by noting the reduction in cost and the change in the terminal values of the rms state deviations and estimation errors at each succeeding pass. In all cases it was found that three forward passes were adequate to achieve a repeatability of 0.1% or better for each of these parameters.

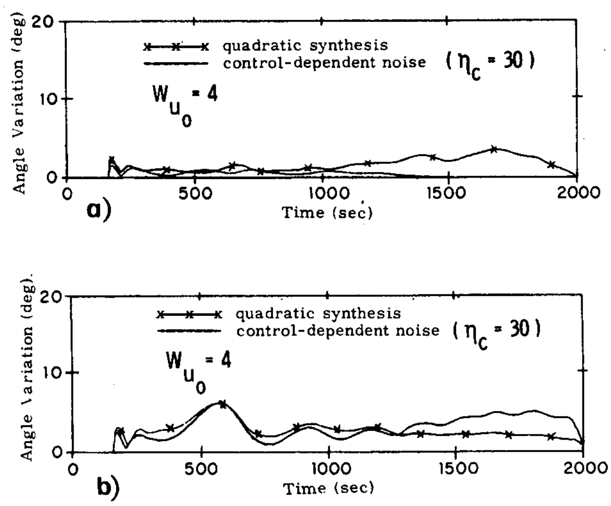

Fig. 3 a) RMS angle-of-attack variation, b) RMS roll angle variation.

Table 1 RMS terminal deviations and estimation errors

Disturbance model	Control weight W_{u_o}	Mult. noise factor η_c/η_s	Control	RMS deviations Alt (ft)	Range (n.mi.)	Speed (fps)	Fl path ang. (deg.)	RMS estimation errors Alt (ft)	Range (n.mi)	Speed (fps)	Fl path angle (deg.)
Additive only	∞	0/0	OL[a]	9530	84.59	628.9	2.01	7214	45.94	483.1	1.45
(quadratic	4		CL	7303	47.24	485.1	1.54				
synthesis	24			7312	47.68	487.0	1.57				
	240			7363	48.51	491.0	1.70				
	2400			7522	50.35	494.9	1.77				
Additive +	4	3/0	CL	7358	47.65	489.1	1.62	7257	46.05	486.4	1.47
control-dependent		10/10		7360	47.83	488.4	1.66	7247	46.08	485.7	1.47
		30/0		7371	48.08	487.7	1.69	7241	46.13	485.2	1.46
Additive + con-	4	30/30	OL	11370	91.52	738.4	2.82	9056	53.28	603.8	2.11
trol-dependent +		30/30	CL	8404	52.05	551.08	2.13	8192	49.90	547.4	1.84
state-dependent		30/3	OL	9668	85.21	637.0	2.07	7349	46.55	491.9	1.50
		30/3	CL	7460	48.46	493.2	1.73	7324	46.48	490.5	1.49

[a] OL = open-loop, CL = closed-loop. All data at $t = 2000$ sec.

C. Numerical Results

An initial set of runs was made to compare the statistical performance with and without multiplicative noise. Accelerometer measurements were not incorporated into the estimate until the altitude dropped below 300,000 ft. The rms terminal state deviations and estimation errors are given in Table 1. The quadratic synthesis (QS) data, found by using the LQG model, was obtained by setting $\eta_s = \eta_c = 0$. For QS, values of $W_{uo} = 4$, 24, 240, 2400 were run. The predicted difference in the rms state deviations were small and were only slightly larger than the estimation errors in each case. With CDN, a value of $W_{u_o} = 4$ was used and η_c assumed values of 3, 10, and 30. For $\eta_c = 30$** $W_{u_o} = 4$, the predicted deviations were only slightly larger than with QS. However, as will be shown later, the control gains were significantly different. Note that, with CDN but no SDN, the filter performance is affected by the controller performance. Altitude, speed, and flight path angle estimation errors are slightly reduced at the higher values of η_c since control activity is reduced here (recall there was no cost for estimation errors). When SDN was included, the open-loop rms deviations increased significantly. The altitude deviation increased from 9530 ft, with $\eta_c = \eta_s = 0$, to 11,370 ft. with $\eta_c = \eta_s = 30$. With closed loop control the predicted errors with $\eta_c = \eta_s = 30$ are not much larger than for $\eta_c = \eta_s = 0$. The estimation errors were significantly reduced when using closed-loop control since the SDN matrices $\Delta(X)$, $\Omega(X)$ are reduced as X is reduced.

The effect of CDN on the predicted rms control activity is depicted in Fig. 3. It can be seen that there are significant differences near the end of the trajectory. The angle-of-attack activity is essentially zero for the CDN design during the final 600 sec, reflecting the uncertainty associated with angle-of-attack control. This is compensated for by increased roll angle activity in the same region. This behavior can be attributed primarily to the effect differences in control weighting assigned throughout the trajectory. From Eq. (30) the effective control weighting with CDN but without SDN is

$$W_{u_{i_{eff}}} = W_{u_i} + f^{NT}_{u_i} \Lambda_{x_i} f^N_{u_i} + \eta_c \bar{\Omega}_i (\Lambda_{x_i} + Y_i^T \Lambda_{p_i} Y_i)$$

where $Y_i = I - K_i h^N_{xi}$. The design factor η_c has been included in the CDN term. A time history of the diagonal elements of $W_{u_{eff}}$ is shown in Fig. 4 for $\eta_c = 30$. By comparison with Fig. 2b it can be seen that for $t < 1200$, the diagonal elements reach local maxima at points corresponding to local maxima of specific force. This implies that control activity should be reduced at these points. Control effectiveness is high, but too much control activity is undesirable due to control uncertainties and trajectory disturbances. Thus an automatic hedge against uncertainty is obtained. Note that effective control weighting on angle-of-attack is much higher throughout than on roll angle. This is due to the uncertainty associated with the effect of angle-of-attack variations on the trajectory. The angle-of-attack is assumed known perfectly but $C_L(\bar{\alpha})$ and $C_D(\bar{\alpha})$ are uncertain. On the other hand, the effect of roll angle variations on the trajectory are known more accurately. In essence angle-of-attack control is being traded for roll control. This is especially true for $t < 1200$. The angle-of-attack weighting, which goes off scale, is between 5000 and 15,000 until the end of the trajectory where it drops rapidly to its minimum value of 4 at $t = 2000$.

The increase in the effective angle-of-attack weighting relative to the weighting on roll can be understood by inspection of the equations of motion, Eqs. (35) and (36). It can be seen that angle-of-attack variations induce uncertainties in both v and γ, due to uncertainties only in $C_D(\bar{\alpha})$ and $C_L(\bar{\alpha})$, whereas roll angle variations induce uncertainties only in γ due to uncertainties in $C_L(\bar{\alpha})$. As the end of the trajectory is approached sin γ increases so that roll can be used more effectively for speed control. The control gains for velocity

**This value was selected based on a limited number of Monte Carlo runs with parameter bias errors.[9]

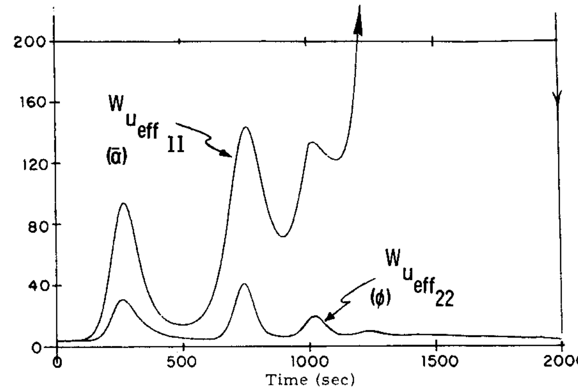

Fig. 4 Diagonal elements of $W_{u_{eff}}$ vs time $\eta_c = 30$, $\eta_s = 0$.

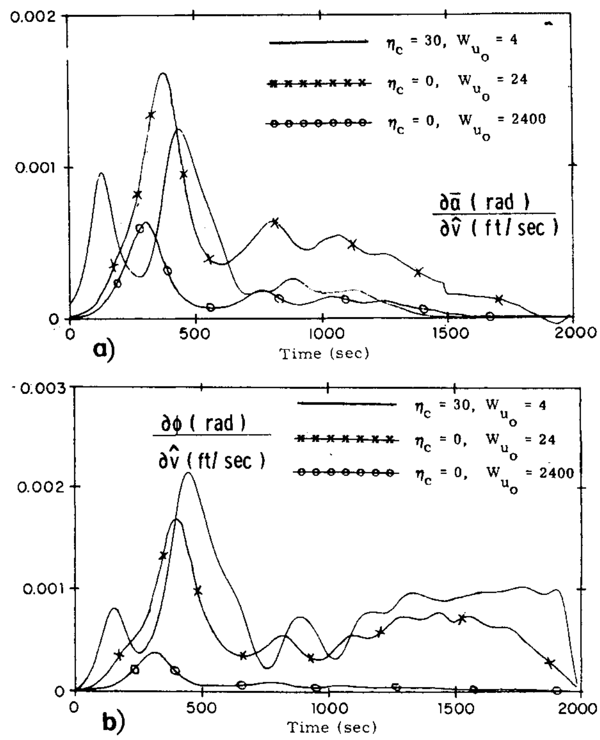

Fig. 5 a) Control gain L_{13} vs time, b) control gain L_{23} vs time.

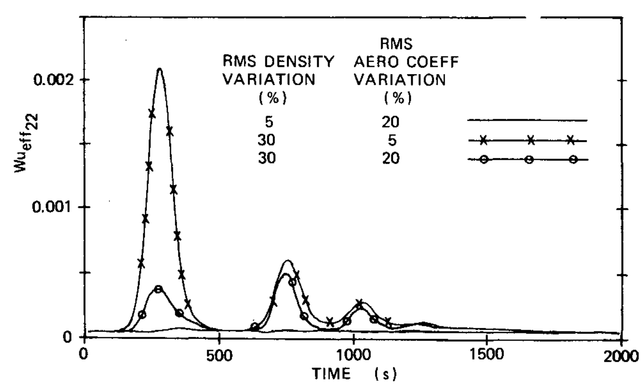

Fig. 6 Effective roll control weighting.

variations are compared for the two designs in Fig. 5. The angle-of-attack gain ($\partial\bar{\alpha}/\partial v$) magnitude decreases while the roll gain ($\partial\phi/\partial\hat{v}$) magnitude increases for the CDN design near the end of the trajectory, when compared to the QS design. The shape of the gain histories is considerably different near the beginning of the trajectory for the two designs. The local minima for the CDN design correspond to the local

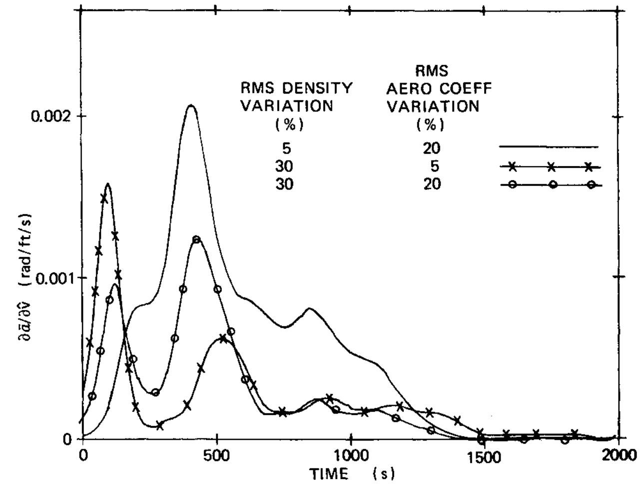

Fig. 7 Control gain ($\partial\bar{\alpha}/\partial\bar{v}$).

maxima of the effective control penalties shown in Fig. 4. The QS gains show no such behavior.

The magnitudes of the rms parameter uncertainties were varied to study their effect on control activity. The results are shown in Figs. 6 and 7. In Fig. 6 it is seen that increased density uncertainty leads to increased roll control weighting initially. Increased aerodynamic coefficient uncertainty suppresses the initial control weighting, while increasing it near the end of the trajectory. The behavior of the angle-of-attack weighting was qualitatively similar. The control gain from velocity variation to angle-of-attack is given in Fig. 7 for the same set of rms parameter uncertainties. For an rms density uncertainty of 30%, the initial peak is suppressed and the second peak is increased as the aerodynamic coefficient uncertainties increase from 5-20%. For an rms uncertainty of 20% in the aerodynamic coefficients, the gain is suppressed dramatically at $t = 300$ (max dynamic pressure) as the density uncertainty is increased from 5-30%.

Conclusions

The theory of combined linear estimation and control of linear systems with white multiplicative noises is applied to the regulation of nonlinear stochastic systems. Certain second-order terms in an expansion about the reference path are modeled as white multiplicative noise in both state and control. The resulting control feedback gains are found to depend on the uncertainty levels of the system parameters; i.e., certainty-equivalence[2] does not hold. In addition, the filter and control gains must be found jointly since the control gains affect the estimator performance; i.e., there is no separation of estimation and control.[3,4] A specific application is made to the guidance/navigation problem for Space Shuttle entry. Numerical results are given to demonstrate the effect of parameter uncertainties on the resulting filter and control gains. An especially interesting aspect of the technique is that the control weighting gains in the quadratic cost functional do not need to be specified bythe designer. Rather, they appear naturally due to the presence of the multiplicative disturbances.

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